S.T. Yau College Student Mathematics Contests 2019
Algebra and Number Theory Individual

1. Consider the group $\mathrm{GL}(m, \mathbb{R})$. Given $g \in \mathrm{GL}(m, \mathbb{R})$, prove there is a decomposition of g as

$$g = k_1 \, d \, k_2$$

where k_1 and k_2 are orthogonal matrices, and d is a diagonal matrix whose diagonal entries are positive.

2. (a) Let $P \in \mathbb{Q}[X]$ be a monic, irreducible polynomial of degree n.

(i) Prove that its roots in $\mathbb{C}$ are simple.

(ii) Prove there exists a matrix $M \in M_{n\times n}(\mathbb{Q})$ whose characteristic polynomial is P.

(b) Let $P \in \mathbb{Z}[X]$ be a monic polynomial of degree n. Prove there exists a matrix $M \in M_{n\times n}(\mathbb{Z})$ whose characteristic polynomial is P and which is diagonalizable over the field $\mathbb{C}$.

3. Prove that a group G of order 48 cannot be a simple group.

4. (a) Describe, in as simple terms as possible, the splitting field of the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

regarded as a matrix with entries in $\mathbb{Q}$. Is A semisimple?

(b) Same problem as (a), but with A now regarded as a matrix with entries in the field $\mathbb{F}_2 =^{\mathrm{def}} \mathbb{Z}/(2\mathbb{Z})$. How many elements does the splitting field have?

(c) Once more the same problem as (a), but over the field $\mathbb{F}_5 =^{\mathrm{def}} \mathbb{Z}/(5\mathbb{Z})$. Is A semisimple now?

(d) Recall the inductive definition of the sequence of the Fibonacci numbers $\{f_n\}$: $f_0 = 1$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$. Argue that the sequence $\{\sqrt[n]{f_n}\}$ must have a limit as $n \to \infty$, and compute that limit.

(e) What does (c) tell you about the behavior of the Fibonacci numbers modulo 5, i.e., as a sequence of values in $\mathbb{F}_5$?

5. Let k an infinite field and take K to be an extension field of k. Let $A, B \in M_{n\times n}(k)$. Prove that if A and B are similar in $M_{n\times n}(K)$, then they are similar in $M_n(k)$.